Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 1, pp. 1 – 16
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.1.1

# On controllability problem with amplitude uncertainty

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ABSTRACT. In this paper, a discrete dynamical system with amplitude uncertainty is considered as an object of interval analysis of the classical interval arithmetic. Within this framework, it is possible to obtain an exact interval analogue of the Cauchy formula for non-negative interval coefficients of the system and to formulate a criterion for non-negative controllability. For arbitrary coefficients of the system, classical interval arithmetic gives an approximate Cauchy formula and, in the final analysis, allows one to obtain sufficient controllability conditions.

2010 MATHEMATICS SABJECT CLASSIFICATION: 34K35, 47N70, 93-XX.

KEYWORDS AND PHRASES. Interval indicator, interval discrete system, Cauchy formula, controllability problem, linear programming problem.

## INTRODUCTION

Problems of control by objects under uncertainty conditions were considered by a number of authors (see, for example, [1, 2, 4-7, 9, 12, 13]). An overview of methods, results, and the main bibliography of papers with interval uncertainties can be found in [4]. Publications [2, 3, 4, 5, 6] are directly devoted to the problem of controllability of interval systems. In [10, 12-16], the problem of stabilization of a controlled linear continuous system with interval coefficients was studied. Under certain conditions on the coefficients, the existence of a deterministic feedback control is proved, which ensures the asymptotic attraction of all trajectories of a closed system to the equilibrium position. The possibility of transferring the results to nonlinear systems is shown. The paper [4] considers a generalization of the classical two-point controllability problem [2−9] to the case of an interval discrete system. The latter is understood as a set of discrete systems whose coefficients take values from given intervals. Such an interpretation allows one to

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circumvent the limitations of classical interval arithmetic, but requires exact estimates of matrix polynomials. Finding exact estimates is rather problematic and the use of approximate estimates inevitably leads to the loss of the necessary controllability conditions. A similar treatment was used in [4] to study the controllability of a continuous linear interval system in the class of piecewise constant bounded controls. The obtained sufficient controllability conditions are formulated in terms of linear programming.

In this paper, an interval discrete system is considered as an object of interval analysis within the framework of classical interval arithmetic. Within this framework, it is possible to obtain an exact interval analogue of the Cauchy formula for non-negative interval coefficients of the system and to formulate a criterion for non-negative controllability. For arbitrary coefficients of the system, classical interval arithmetic gives an approximate Cauchy formula (external interval estimate of the trajectory) and, in the final analysis, allows one to obtain only sufficient controllability conditions.

## 1. PRELIMINARIES

By [11], we consider the space $\mathbb{IR}$ of regular closed real intervals $\boldsymbol{a}=[\underline{a},\overline{a}],\ \underline{a}\le\overline{a}$ . The *center* and *radius* of the interval $\boldsymbol{a}$ will be denoted

$$a_0=0.5(\underline{a}+\overline{a}),\ \Delta a=0.5(\overline{a}-\underline{a}).$$

Expressing the ends of the interval in terms of the center and the radius, we obtain an equivalent *symmetrical* representation of the interval

$$\boldsymbol{a}=[a_0-\Delta a, a_0+\Delta a].$$

An interval $\boldsymbol{a}$ is called *degenerate* if $\Delta a=0$ and *non-degenerate* if $\Delta a>0$.

Following [17], for intervals $\boldsymbol{a},\boldsymbol{b}\in\mathbb{IR}$ we give the definitions of inequality $\boldsymbol{a}\le\boldsymbol{b}$ in the «strong», «weak» and «central» senses:

$$\begin{gathered}\boldsymbol{a}\le\boldsymbol{b}\Leftrightarrow\big((\forall a\in\boldsymbol{a})(\forall b\in\boldsymbol{b})(a\le b)\big),\\ \boldsymbol{a}\le\boldsymbol{b}\Leftrightarrow\big((\exists a\in\boldsymbol{a})(\exists b\in\boldsymbol{b})(a\le b)\big),\\ \boldsymbol{a}\le\boldsymbol{b}\Leftrightarrow(a_0\le b_0).\end{gathered}\tag{1.1}$$

and accept the real number

$$R(\boldsymbol{a} \le \boldsymbol{b}) = \frac{b_0 - a_0}{\Delta a + \Delta b} \tag{1.2}$$

as the *indicator* $R$ *of the interval inequality* $\boldsymbol{a} \le \boldsymbol{b}$.

By [11], for intervals $\boldsymbol{a}, \boldsymbol{b}$ from $\mathbb{IR}$ represented in symmetrical form

$$\boldsymbol{a} = [a_0 - \Delta a, a_0 + \Delta a],\ \boldsymbol{b} = [b_0 - \Delta b, b_0 + \Delta b],$$

operations of classical interval arithmetic (addition, multiplication by a real number $\alpha$) are performed according to the rules

$$\boldsymbol{a} + \boldsymbol{b} = [a_0 + b_0 - \Delta a - \Delta b,\ a_0 + b_0 + \Delta a + \Delta b],$$

$$\alpha \boldsymbol{a} = [\alpha a_0 - |\alpha| \Delta a,\ \alpha a_0 + |\alpha| \Delta a].$$

Using these operations, it is easy to establish [17] the main properties of the indicator which follow from definition (1.2).

- Multiplying the interval inequality by a positive number does not change the inequality indicator; multiplying inequality by a negative number reverses the sign of the indicator.
- Inequality indicator is antisymmetric: $R(\boldsymbol{a} \le \boldsymbol{b}) = -R(\boldsymbol{b} \le \boldsymbol{a})$.
- Intervals $\boldsymbol{a}$, $\boldsymbol{b}$ with equal centers satisfy opposite inequalities $\boldsymbol{a} \le \boldsymbol{b}, \boldsymbol{b} \le \boldsymbol{a}$ with zero indicator.
- When adding interval inequalities $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{c} \le \boldsymbol{d}$ with equal indicators, an inequality $\boldsymbol{a} + \boldsymbol{c} \le \boldsymbol{b} + \boldsymbol{d}$ with the same indicator is obtained.
- If pairs of intervals $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, $\boldsymbol{d}$ have equal sums of radii then the inequality indicator $\boldsymbol{a} + \boldsymbol{c} \le \boldsymbol{b} + \boldsymbol{d}$ is equal to the arithmetic mean of the inequality indicators $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{c} \le \boldsymbol{d}$.
- For intervals $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ with pairwise equal positive sums of radii, the equality

$$R(\boldsymbol{a} \le \boldsymbol{c}) = R(\boldsymbol{a} \le \boldsymbol{b}) + R(\boldsymbol{b} \le \boldsymbol{c})$$

holds true.

## 2. THE CONTROLLABILITY PROBLEM

***2.1. Discrete system. Non-negativity of a solution***

Consider a linear interval *discrete system*

$$\boldsymbol{x}^{(t+1)} = \boldsymbol{A}^{(t)}\boldsymbol{x}^{(t)} + \boldsymbol{b}^{(t)},\ t = \tau, \tau+1, \ldots;\ \ \boldsymbol{x}^{(\tau)} = \boldsymbol{\xi}\,, \tag{2.1}$$

where $t$ is integer variable, $\boldsymbol{A}^{(t)} \in \mathrm{IR}^{n\times n}$, $\boldsymbol{b}^{(t)}$, $\boldsymbol{\xi} \in \mathrm{IR}^{n}$ are known interval matrices and vectors, $\boldsymbol{x}^{(t)} \in \mathrm{IR}^{n}$ are required interval vectors. A *solution* or a *trajectory* of a discrete system is a sequence of interval vectors

$$\boldsymbol{x}^{(\tau)} = \boldsymbol{\xi},\ \boldsymbol{x}^{(\tau+1)} = \boldsymbol{A}^{(\tau)}\boldsymbol{x}^{(\tau)} + \boldsymbol{b}^{(\tau)},\ \boldsymbol{x}^{(\tau+2)} = \boldsymbol{A}^{(\tau+1)}\boldsymbol{x}^{(\tau+1)} + \boldsymbol{b}^{(\tau+1)}, \ldots. \tag{2.2}$$

We need a criterion for the non-negativity of a solution of the system (2.1). Let us call a matrix $\boldsymbol{C} \in \mathrm{IR}^{m\times n}$ *non-negative* and write $\boldsymbol{C} \geq 0$ if all its elements lie in a half-segment $[0,\infty)$. Solution (2.2) of a discrete system is considered *non-negative* if the inequalities

$$\boldsymbol{x}^{(t)} \geq 0,\ t \geq \tau\,, \tag{2.3}$$

hold.

We establish a criterion for non-negativity of a solution under the assumptions

$$\boldsymbol{A}^{(t)} \geq 0, t \geq \tau,\ \boldsymbol{\xi} \geq 0\,. \tag{2.4}$$

Let the solution (2.2) of the discrete system be non-negative. We choose any $t \geq \tau$ and write recursion (2.1) in coordinate form

$$\boldsymbol{x}_i^{(t+1)} = \sum_{j=1}^{n} \boldsymbol{a}_{ij}^{(t)}\boldsymbol{x}_j^{(t)} + \boldsymbol{b}_i^{(t)},\ i = 1,\ldots,n\,.$$

In the detailed notations, based on the assumptions (2.3), (2.4), we have

$$[\underline{x}_i^{(t+1)}, \overline{x}_i^{(t+1)}] = \sum_{j=1}^{n} [\underline{a}_{ij}^{(t)}, \overline{a}_{ij}^{(t)}][\underline{x}_j^{(t)}, \overline{x}_j^{(t)}] + [\underline{b}_i^{(t)}, \overline{b}_i^{(t)}] =$$

$$= \sum_{j=1}^{n} [\underline{a}_{ij}^{(t)}\underline{x}_j^{(t)}, \overline{a}_{ij}^{(t)}\overline{x}_j^{(t)}] + [\underline{b}_i^{(t)}, \overline{b}_i^{(t)}] =$$

$$=\left[\sum_{j=1}^{n}\underline{a}_{ij}^{(t)}\underline{x}_j^{(t)}+\underline{b}_i^{(t)},\sum_{j=1}^{n}\overline{a}_{ij}^{(t)}\overline{x}_j^{(t)}+\overline{b}_i^{(t)},\right]$$

$$i=1,...,n.$$

Consequently,

$$\underline{x}_i^{(t+1)}=\sum_{j=1}^{n}\underline{a}_{ij}^{(t)}\underline{x}_j^{(t)}+\underline{b}_i^{(t)}\geq 0;\ \underline{x}_i^{(\tau)}=\underline{\xi}_i\geq 0,\ i=1,...,n,$$

or in vector-matrix form

$$\begin{aligned}&\underline{x}^{(t+1)}=\underline{A}^{(t)}\underline{x}^{(t)}+\underline{b}^{(t)}\geq 0,\ t\geq\tau,\\&\underline{x}^{(\tau)}=\underline{\xi}\geq 0.\end{aligned}\tag{2.5}$$

Conditions (2.5) are necessary for non-negativity of the solution of the discrete system. By doing the arguments in reverse order, we will see that they are sufficient. We summarize.

**Lemma 2.1**. *If assumptions (2.4) hold then the interval discrete system (2.1) has a non-negative solution if and only if conditions (2.5) are satisfied.*

***Example 2.1***

Let matrix $A^{(t)}=[-\varepsilon A,\varepsilon A],\ A\geq 0$, and vector $b^{(t)}=b$ be constant, $\varepsilon$ be a small positive parameter. We find out under what conditions on $A,b,\xi$ the first terms of a solution of discrete system are non-negative. By formulas (2.5), the criterion for non-negativity of a solution takes the form

$$\begin{aligned}&\underline{x}^{(\tau)}=\underline{\xi}\geq 0,\\&\underline{x}^{(\tau+1)}=-\varepsilon A\underline{\xi}+b\geq 0,\\&\underline{x}^{(\tau+2)}=...-\varepsilon Ab+b\geq 0,\\&\underline{x}^{(\tau+3)}=...-\varepsilon Ab+b\geq 0,\\&.....................................\end{aligned}$$

where the ellipsis denotes terms of order higher than $\varepsilon$. This shows that due to the smallness $\varepsilon$ the first terms of the sequence (2.2) are non-negative if $\xi\geq 0,\ b>0$.

### *2.2. Cauchy formula*

The purpose of this section is to represent the solution of a discrete system as a known function of its coefficients and initial state. Let us first show that for non-negative intervals $\boldsymbol{a},\boldsymbol{u},\boldsymbol{v}\in \mathrm{IR}$ the usual rule of taking the common factor out of brackets takes place, i.e.

$$\boldsymbol{au}+\boldsymbol{av}=\boldsymbol{a}(\boldsymbol{u}+\boldsymbol{v}).$$

Indeed, according to the rules of interval arithmetic, by virtue of non-negativity of intervals, we have

$$\boldsymbol{au}=[\underline{a},\overline{a}][\underline{u},\overline{u}]=[\min\{\underline{a}\,\underline{u},\underline{a}\,\overline{u},\overline{a}\,\underline{u},\overline{a}\,\overline{u}\},\\ \max\{\underline{a}\,\underline{u},\underline{a}\,\overline{u},\overline{a}\,\underline{u},\overline{a}\,\overline{u}\}]=[\underline{a}\,\underline{u},\overline{a}\,\overline{u}],$$

therefore

$$\boldsymbol{au}+\boldsymbol{av}=[\underline{a}\,\underline{u},\overline{a}\,\overline{u}]+[\underline{a}\,\underline{v},\overline{a}\,\overline{v}]=\\ =[\underline{a}(\underline{u}+\underline{v}),\ \overline{a}(\overline{u}+\overline{v})]=\boldsymbol{a}(\boldsymbol{u}+\boldsymbol{v}).$$

The established rule is easily generalized to the case of non-negative real interval matrices $\boldsymbol{A}$, $\boldsymbol{U}$, $\boldsymbol{V}$ of consistent dimension:

$$\boldsymbol{AU}+\boldsymbol{AV}=\boldsymbol{A}(\boldsymbol{U}+\boldsymbol{V}). \tag{2.6}$$

Let us return to solution (2.2) of the interval discrete system (2.1). For a compact representation of the solution, we introduce a matrix $\boldsymbol{F}^{(t,\tau)}\in \mathrm{IR}^{n\times n}$ that satisfies the conditions

$$\begin{aligned}&\boldsymbol{F}^{(t+1,\tau)}=\boldsymbol{A}^{(t)}\boldsymbol{F}^{(t,\tau)},\ t\geq\tau,\\ &\boldsymbol{F}^{(\tau,\tau)}=E,\ \boldsymbol{F}^{(t,\tau)}=0,\ t<\tau,\end{aligned} \tag{2.7}$$

where $E,0\in \mathrm{R}^{n\times n}$ are the identity and zero matrices.

**Lemma 2.2**. *If*

$$\boldsymbol{A}^{(t)}\geq 0,\boldsymbol{b}^{(t)}\geq 0, t\geq\tau,\ \boldsymbol{\xi}\geq 0, \tag{2.8}$$

*then the unique solution of the discrete system (2.1) can be represented in the form*

$$\boldsymbol{x}^{(t)}=\boldsymbol{F}^{(t,\tau)}\boldsymbol{\xi}+\sum_{s=\tau}^{t-1}\boldsymbol{F}^{(t,s+1)}\boldsymbol{b}^{(s)},\ t\geq\tau. \tag{2.9}$$

We prove the assertion by the method of mathematical induction. Let assumptions (2.8) be satisfied. For $t=\tau$, the validity of formula (2.9) follows from the last conditions in (2.7). Suppose equality (2.9) is true for some $t>\tau$. Replacing in it *t* by $t+1$, we get

$$\boldsymbol{x}^{(t+1)} = \boldsymbol{F}^{(t+1,\tau)}\boldsymbol{\xi} + \sum_{s=\tau}^{t} \boldsymbol{F}^{(t+1,s+1)}\boldsymbol{b}^{(s)} =$$

$$= \boldsymbol{A}^{(t)}\boldsymbol{F}^{(t,\tau)}\boldsymbol{\xi} + \sum_{s=\tau}^{t-1} \boldsymbol{A}^{(t)}\boldsymbol{F}^{(t,s+1)}\boldsymbol{b}^{(s)} + \boldsymbol{b}^{(t)}.$$

Since the factors at $\boldsymbol{A}^{(t)}$ are non-negative vectors then on the basis of equality (2.6) we have

$$\boldsymbol{x}^{(t+1)} = \boldsymbol{A}^{(t)}\boldsymbol{F}^{(t,\tau)}\boldsymbol{\xi} + \sum_{s=\tau}^{t-1} \boldsymbol{A}^{(t)}\boldsymbol{F}^{(t,s+1)}\boldsymbol{b}^{(s)} + \boldsymbol{b}^{(t)} =$$

$$= \boldsymbol{A}^{(t)}\boldsymbol{F}^{(t,\tau)}\boldsymbol{\xi} + \boldsymbol{A}^{(t)}\sum_{s=\tau}^{t-1} \boldsymbol{F}^{(t,s+1)}\boldsymbol{b}^{(s)} + \boldsymbol{b}^{(t)} =$$

$$= \boldsymbol{A}^{(t)}\left[\boldsymbol{F}^{(t,\tau)}\boldsymbol{\xi} + \sum_{s=\tau}^{t-1} \boldsymbol{F}^{(t,s+1)}\boldsymbol{b}^{(s)}\right] + \boldsymbol{b}^{(t)} = \boldsymbol{A}^{(t)}\boldsymbol{x}^{(t)} + \boldsymbol{b}^{(t)}.$$

Consequently, formula (2.9) gives the same sequence of vectors (2.2) which is a solution to the discrete system (2.1). Arithmetic operations determine the interval vectors (2.2) uniquely, so the solution of the discrete system is unique. The assertion has been proven.

Formula (2.9) is called the *Cauchy formula* and the matrix $\boldsymbol{F}^{(t,\tau)}$ is called the *fundamental matrix*. As can be seen, the Cauchy formula represents the solution of a discrete system directly in terms of its coefficients and initial state.

The Cauchy formula can be simplified if in the discrete system

$$\boldsymbol{x}^{(t+1)} = \boldsymbol{A}\boldsymbol{x}^{(t)} + \boldsymbol{b}^{(t)},\ t=0,1,...;\ \ \boldsymbol{x}^{(0)} = \boldsymbol{\xi}, \tag{2.10}$$

matrix $\boldsymbol{A}$ is constant. By virtue of (2.7), we then obtain $\boldsymbol{F}^{(t,0)} = \boldsymbol{A}^{t},\ t>\tau$, and under the conditions

$$\boldsymbol{A} \geq 0, \boldsymbol{b}^{(t)} \geq 0, t \geq \tau,\ \ \boldsymbol{\xi} \geq 0 \tag{2.11}$$

a solution of the system (2.10) takes the form

$$\boldsymbol{x}^{(t)} = \boldsymbol{A}^t \boldsymbol{\xi} + \sum_{s=0}^{t-1} \boldsymbol{A}^{t-s-1} \boldsymbol{b}^{(s)},\ t \geq 0. \tag{2.12}$$

Note that for a non-interval discrete system

$$x^{(t+1)} = A^{(t)} x^{(t)} + b^{(t)},\ t = \tau, \tau+1, \ldots;\ \ x^{(\tau)} = \xi, \tag{2.13}$$

Cauchy formula

$$x^{(t)} = F^{(t,\tau)} \xi + \sum_{s=\tau}^{t-1} F^{(t,s+1)} b^{(s)},\ t \geq \tau, \tag{2.14}$$

with fundamental matrix

$$\begin{aligned} &F^{(t+1,\tau)} = A^{(t)} F^{(t,\tau)},\ t \geq \tau, \\ &F^{(\tau,\tau)} = E,\ F^{(t,\tau)} = 0,\ t < \tau, \end{aligned} \tag{2.15}$$

is valid without the assumptions about non-negativity of $A^{(t)}, b^{(t)}, \xi$. In particular, if the matrix $A^{(t)} = A$ is constant then from (2.14), (2.15) it follows

$$x^{(t)} = A^{t-\tau} \xi + \sum_{s=\tau}^{t-1} A^{t-s-1} b^{(s)},\ t \geq 0. \tag{2.16}$$

### *2.3. Non-negative controllability of a discrete system*

Consider a controlled interval discrete system

$$\boldsymbol{x}^{(t+1)} = \boldsymbol{A}^{(t)} \boldsymbol{x}^{(t)} + \boldsymbol{B}^{(t)} u^{(t)},\ \boldsymbol{x}^{(t_0)} = \boldsymbol{x}_0,\ t \geq t_0, \tag{2.17}$$

where $\boldsymbol{A}^{(t)} \in \mathrm{IR}^{n\times n}$, $\boldsymbol{B}^{(t)} \in \mathrm{IR}^{n\times m}, t \geq t_0$, are given matrices, $\boldsymbol{x}_0 \in \mathrm{IR}^n$ is given vector, $\boldsymbol{x}^{(t)} \in \mathrm{IR}^n$ are *state* vectors, $u^{(t)} \in \mathrm{R}^m$ are control actions.

We call the sequence of vectors $u^{(t)} \in [-1,1]^m \subset \mathrm{R}^m, t \geq t_0$, the *control*, the sequence of the corresponding state vectors $\boldsymbol{x}^{(t)}, t \geq t_0$, of the system (2.17) – a *trajectory*, and the pair $\boldsymbol{x}^{(t)}, u^{(t)}$ – a process. System (2.17) is said to be *non-negatively controllable* to position $\boldsymbol{x}_1, t_1$ if there exists a process $\boldsymbol{x}^{(t)}, \boldsymbol{u}^{(t)}$ satisfying the conditions

$$\boldsymbol{x}^{(t)} \geq 0,\ t = t_0, \ldots, t_1 - 1;\ \boldsymbol{x}^{(t_1)} \subset \boldsymbol{x}_1. \tag{2.18}$$

Here $t_1, t_1 > t_0$, is a given natural number, $\boldsymbol{x}_1 \in \mathrm{IR}^n, \boldsymbol{x}_1 \geq 0$, is a given vector. A process with properties (2.18) is called *admissible*.

We establish a criterion for the existence of an admissible process under the assumption $\boldsymbol{A}^{(t)} \geq 0, t \geq t_0$. Let $\boldsymbol{x}^{(t)}, u^{(t)}$ be an admissible process of the system (2.17) satisfying conditions (2.18). Then the following relations are true

$$\begin{aligned} &\boldsymbol{x}^{(t+1)} = \boldsymbol{A}^{(t)}\boldsymbol{x}^{(t)} + \boldsymbol{B}^{(t)}u^{(t)} \geq 0,\ t = t_0,\ldots,t_1 - 1, \\ &\boldsymbol{x}_0 \geq 0,\ \boldsymbol{x}^{(t_1)} \subset \boldsymbol{x}_1. \end{aligned} \tag{2.19}$$

Define the boundaries of the interval vector $\boldsymbol{x}^{(t+1)}$ in (2.19). By analogy with Section 2.1, we find

$$\boldsymbol{A}^{(t)}\boldsymbol{x}^{(t)} = [\underline{A}^{(t)}\underline{x}^{(t)}, \overline{A}^{(t)}\overline{x}^{(t)}]. \tag{2.20}$$

For an interval vector $\boldsymbol{b}^{(t)} = \boldsymbol{B}^{(t)}u^{(t)}$ in coordinate form, we get

$$\begin{aligned} \boldsymbol{b}_i^{(t)} &= \sum_{j=1}^{m} \boldsymbol{b}_{ij}^{(t)} u_j^{(t)} = \sum_{j=1}^{m} [b_{ij0}^{(t)} - \Delta b_{ij}^{(t)}, b_{ij0}^{(t)} + \Delta b_{ij}^{(t)}] u_j^{(t)} = \\ &= \sum_{j=1}^{m} \left[ b_{ij0}^{(t)} u_j^{(t)} - \Delta b_{ij}^{(t)} \left| u_j^{(t)} \right|,\ b_{ij0}^{(t)} u_j^{(t)} + \Delta b_{ij}^{(t)} \left| u_j^{(t)} \right| \right] = \\ &\left[ \sum_{j=1}^{m} b_{ij0}^{(t)} u_j^{(t)} - \sum_{j=1}^{n} \Delta b_{ij}^{(t)} \left| u_j^{(t)} \right|,\ \sum_{j=1}^{m} b_{ij0}^{(t)} u_j^{(t)} - \sum_{j=1}^{n} \Delta b_{ij}^{(t)} \left| u_j^{(t)} \right| \right], \\ &\qquad i = 1,\ldots,n. \end{aligned}$$

Passing to the vector-matrix notation, we have

$$\boldsymbol{b}^{(t)} = \boldsymbol{B}^{(t)}u^{(t)} = \left[ B_0^{(t)}u^{(t)} - \Delta B^{(t)} \left| u^{(t)} \right|,\ B_0^{(t)}u^{(t)} + \Delta B^{(t)} \left| u^{(t)} \right| \right], \tag{2.21}$$

where $\left| u^{(t)} \right|$ is the vector formed by the modules of the coordinates of the vector $u^{(t)}$. Substituting (2.20), (2.21) into (2.19), we find

$$\begin{aligned} \boldsymbol{x}^{(t+1)} &= [\underline{x}^{(t+1)}, \overline{x}^{(t+1)}] = \left[ \underline{A}^{(t)}\underline{x}^{(t)} + B_0^{(t)}u^{(t)} - \Delta B^{(t)} \left| u^{(t)} \right|, \right. \\ &\left. \overline{A}^{(t)}\overline{x}^{(t)} + B_0^{(t)}u^{(t)} + \Delta B^{(t)} \left| u^{(t)} \right| \right] \geq 0,\ t = t_0,\ldots,t_1 - 1. \end{aligned} \tag{2.22}$$

Taking into account formulae (2.22), the necessary conditions (2.19) for the existence of an admissible process take the form

$$\underline{x}^{(t)} \geq 0,\ t = t_0 + 1, \ldots, t_1 - 1,$$
$$\underline{x}_0 \geq 0,\ \underline{x}^{(t_1)} \geq \underline{x}_1,\ \ \overline{x}^{(t_1)} \leq \overline{x}_1, \tag{2.23}$$

where

$$\underline{x}^{(t+1)} = \underline{A}^{(t)}\underline{x}^{(t)} + B_0^{(t)}u^{(t)} - \Delta B^{(t)}\left|u^{(t)}\right|,\ \underline{x}^{(t_0)} = \underline{x}_0,$$
$$\overline{x}^{(t+1)} = \overline{A}^{(t)}\overline{x}^{(t)} + B_0^{(t)}u^{(t)} + \Delta B^{(t)}\left|u^{(t)}\right|,\ \overline{x}^{(t_0)} = \overline{x}_0, \tag{2.24}$$
$$t = t_0, \ldots, t_1 - 1.$$

Reversing the reasoning we come to the conclusion that conditions (2.23), (2.24) are sufficient for the admissibility of the process $\boldsymbol{x}^{(t)}, u^{(t)}$.

Using relations (2.13) - (2.15) we express the solutions of non-interval discrete systems (2.24) applying the Cauchy formula. We get

$$\underline{x}^{(t)} = \underline{F}^{(t,t_0)}\underline{x}_0 + \sum_{s=t_0}^{t-1}\underline{F}^{(t,s+1)}\left(B_0^{(s)}u^{(s)} - \Delta B^{(s)}\left|u^{(s)}\right|\right),$$
$$\overline{x}^{(t)} = \overline{F}^{(t,t_0)}\overline{x}_0 + \sum_{s=t_0}^{t-1}\overline{F}^{(t,s+1)}\left(B_0^{(s)}u^{(s)} + \Delta B^{(s)}\left|u^{(s)}\right|\right), \tag{2.25}$$
$$t = t_0, \ldots, t_1.$$

Here the fundamental matrices are defined by the conditions

$$\underline{F}^{(t+1,t_0)} = A^{(t)}\underline{F}^{(t,t_0)},\ t \geq t_0, \underline{F}^{(t_0,t_0)} = E,\ \ \underline{F}^{(t,t_0)} = 0,\ t < t_0;$$
$$\overline{F}^{(t+1,t_0)} = A^{(t)}\overline{F}^{(t,t_0)},\ t \geq t_0, \overline{F}^{(t_0,t_0)} = E,\ \ \overline{F}^{(t,t_0)} = 0,\ t < t_0. \tag{2.26}$$

Substituting (2.25) into (2.23) gives the admissibility criterion of a process expressed in terms of control:

$$
\begin{aligned}
&\underline{x}_0 \geq 0,\\
&\underline{F}^{(t,t_0)}\underline{x}_0 + \sum_{s=t_0}^{t-1} \underline{F}^{(t,s+1)}\left(B_0^{(s)}u^{(s)} - \Delta B^{(s)}\left|u^{(s)}\right|\right) \geq 0,\ t = t_0 + 1,\dots,t_1 - 1,\\
&\underline{F}^{(t_1,t_0)}\underline{x}_0 + \sum_{s=t_0}^{t_1-1} \underline{F}^{(t_1,s+1)}\left(B_0^{(s)}u^{(s)} - \Delta B^{(s)}\left|u^{(s)}\right|\right) \geq \underline{x}_1,\\
&\overline{F}^{(t_1,t_0)}\overline{x}_0 + \sum_{s=t_0}^{t_1-1} \overline{F}^{(t_1,s+1)}\left(B_0^{(s)}u^{(s)} + \Delta B^{(s)}\left|u^{(s)}\right|\right) \leq \overline{x}_1.
\end{aligned}
\tag{2.27}
$$

We summarize.

**Theorem 2.1**. *Let the matrices* $\boldsymbol{A}^{(t)}, t \geq t_0$, *in the discrete system (2.15) be nonnegative. For non-negative controllability of the system to position* $\boldsymbol{x}_1, t_1$, *it is necessary and sufficient that the initial vector* $\boldsymbol{x}_0$ *and control actions* $u^{(t_0)},\dots,u^{(t_1-1)}$ *from the cube* $[-1,1]^m$ *satisfy inequalities (2.27) in which the fundamental matrices are defined by conditions (2.26).*

Relations (2.27) are a system of non-linear inequalities with concave and convex piecewise linear left-hand sides with respect to the vectors $u^{(t_0)},\dots,u^{(t_1-1)}$. If conditions (2.27) are compatible then they define a convex polyhedron in space $\mathrm{R}^{m(t_1-t_0)}$. For the system (2.17) with non-interval matrix coefficients, conditions (2.27) become a system of linear inequalities.

Linear programming serves as an effective means of checking the criterion (2.27). Using conditions (2.27), we compose the linear programming problem

$$\sum_{s=t_0}^{t_1-1} e' v^{(s)} \to \min,$$

$$\underline{F}^{(t,t_0)} \underline{x}_0 + \sum_{s=t_0}^{t-1} \underline{F}^{(t,s+1)} \left( B_0^{(s)} u^{(s)} - \Delta B^{(s)} v^{(s)} \right) \geq 0,$$

$$t = t_0 + 1, \ldots, t_1 - 1,$$

$$\underline{F}^{(t_1,t_0)} \underline{x}_0 + \sum_{s=t_0}^{t_1-1} \underline{F}^{(t_1,s+1)} \left( B_0^{(s)} u^{(s)} - \Delta B^{(s)} v^{(s)} \right) \geq \underline{x}_1,$$

$$\overline{F}^{(t_1,t_0)} \overline{x}_0 + \sum_{s=t_0}^{t_1-1} \overline{F}^{(t_1,s+1)} \left( B_0^{(s)} u^{(s)} + \Delta B^{(s)} v^{(s)} \right) \leq \overline{x}_1.$$

(2.28)

$$-v^{(t)} \leq u^{(t)} \leq v^{(t)}, \; 0 \leq v^{(t)} \leq e, \; t = t_0, \ldots, t_1 - 1,$$

with unknowns $u^{(0)}, \ldots, u^{(t_1-1)}, v^{(0)}, \ldots, v^{(t_1-1)}$ and $e = (1, \ldots, 1) \in \mathrm{R}^m$.

It is easy to see that the compatibility of inequalities (2.27) implies the compatibility of inequalities (2.28). In this case, due to the boundedness from below of the objective function (2.28), the linear programming problem has the solution

$$u^{(t_0)}, \ldots, u^{(t_1-1)}, v^{(t_0)} = \left| u^{(t_0)} \right|, \ldots, v^{(t_1-1)} = \left| u^{(t_1-1)} \right|.$$

This solution gives the desired sequence of control actions $u^{(t_0)}, \ldots, u^{(t_1-1)}$, ensuring non-negative controllability of the system (2.15) in the position $\boldsymbol{x}_1, t_1$. If the linear programming problem has no solutions due to the inconsistency of constraints then the criterion (2.27) is not fulfilled.

***Example 2.2***

Let us show the application of Theorem 2.1 on the model example

$$\boldsymbol{y}_1((t+1)h) = \boldsymbol{y}_1(th) + h\boldsymbol{y}_2(th),$$

$$\boldsymbol{y}_2((t+1)h) = \boldsymbol{y}_2(th) + (h/\boldsymbol{m}) f, \; t = 0, 1, \ldots.$$

For a small $h = 1/t_1 > 0$, these equations are a difference approximation of differential equations $\dot{y}_1 = y_2$, $\dot{y}_2 = f/m$ describing the rectilinear motion of a material point of unknown mass $m \in \boldsymbol{m} = [1/3, 1]$ under the action of a constant force $f$ without taking into account the resistance of the medium. Assuming

$$\boldsymbol{y}_1(th) = \boldsymbol{x}_1^{(t)},\ \boldsymbol{y}_2(th) = \boldsymbol{x}_2^{(t)},\ f = u,\ 1/\boldsymbol{m} = \boldsymbol{b} = [1,3],$$

we represent the conditions of the example in standard form

$$\begin{aligned} \boldsymbol{x}_1^{(t+1)} &= \boldsymbol{x}_1^{(t)} + h\boldsymbol{x}_2^{(t)}, \\ \boldsymbol{x}_2^{(t+1)} &= \boldsymbol{x}_2^{(t)} + h\boldsymbol{b}u,\ t = 0,1,\dots . \end{aligned} \tag{2.29}$$

We clarify non-negative controllability of the system (2.29) from the initial position $x_0 = (0,0),\ t_0 = 0$, to the position $\boldsymbol{x}_1 = ([0,2],[1,3]),\ t = t_1$, with constant control $u$. For matrix coefficients

$$A = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 0 \\ h\boldsymbol{b} \end{pmatrix}$$

of the system (2.29) using (2.26) we find

$$F^{(t,t_0)} = \underline{F}^{(t,t_0)} = \overline{F}^{(t,t_0)} = \begin{pmatrix} 1 & (t-t_0)h \\ 0 & 1 \end{pmatrix},$$

$$\sum_{s=0}^{t-1} F^{(t,s+1)} = t\begin{pmatrix} 1 & (t-1)h/2 \\ 0 & 1 \end{pmatrix},\ t \ge 1,$$

$$B_0 u \pm \Delta B|u| = h\begin{pmatrix} 0 \\ 2u \pm |u| \end{pmatrix}.$$

As a result, the criterion (2.27) for the data of the example and $t_1 h = 1$ takes the form

$$\begin{aligned} &2u - |u| \ge 0,\ \ 2u - |u| \ge 1, \\ &(1-h)(2u + |u|) \le 4,\ \ (2u + |u|) \le 3,\ \ |u| \le 1. \end{aligned} \tag{2.30}$$

Inequalities (2.30) are satisfied by the unique control $u = 1$. This control ensures non-negative controllability of the system (2.29) from one given position to another in $t_1$ steps.

In the end, we note that interval analysis serves as a tool for studying the problem of controllability. Its visual means and technical capabilities are used to formulate the problem, carry out calculations and present results. This ensures the integrity of the approach to the problem and, at the same time, affects the generality of the results. For example, the rules of interval arithmetic make it possible to obtain an exact interval analogue of the Cauchy formula only under the additional assumption that the solution of the system of discrete equations is nonnegative and the matrix of coefficients of its homogeneous part is nonnegative. For such systems, using the Cauchy formula, a criterion for non-negative controllability is established in terms of the solvability of a finite system of inequalities with modular nonlinearities. By means of linear programming, one can check the solvability of this system of inequalities and find an admissible control if it exists.

## CONCLUSION

In this paper, we considered an interval discrete system as an object of interval analysis within the framework of classical interval arithmetic and obtained an exact interval analogue of the Cauchy formula for non-negative interval coefficients of the system as well as formulated a criterion for non-negative controllability (Theorem 2.1). For interval coefficients of the system, classical interval arithmetic gives an approximate Cauchy formula (external interval estimate of the trajectory) and, in the final analysis, allows one to obtain only sufficient controllability conditions.

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